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# Microwave billiards with broken time reversal invariance

Fritz Haake†, Marek Kuś†‖, Petr Šeba‡, Hans-Jürgen Stöckmann§ and Ulrich Stoffregen§

† Fachbereich Physik, Universität GH Essen, 45117 Essen, Germany
‡ Nuclear Physics Institute, Czech Academy of Sciences, 250 68 Řež, near Prague, Czech Republic
§ Fachbereich Physik, Universität Marburg, D-35032 Marburg, Germany

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**Abstract.** We consider a microwave resonator with three single-channel waveguides attached. One of these serves to couple waves into and out of the resonator; the remaining two are connected to form a one-way handle so as to break time reversal invariance. The poles of the input–output scattering coefficient of such a resonator are shown to be the eigenvalues of a non-Hermitian effective 'Hamiltonian' $H_{\mathrm{eff}} = H - \mathrm{i}\Gamma$, the anti-Hermitian part $\Gamma$ of which has rank 1 and is responsible for the breaking of time reversal invariance. All of the spectral statistics recently observed for such a microwave billiard are reproduced quantitatively by taking $H$ and $\Gamma$ as random matrices. In particular, the distribution of nearest-neighbour spacings of the resonances is close to that of the GUE when $H$ belongs to the GOE corresponding to a Sinai shape of the resonator; linear level repulsion results when $H$ belongs to the Poissonian ensemble as it corresponds to a rectangular resonator.

## 1. Introduction

Already in the 1960s Wigner and others noticed that the spectra of nuclei can be excellently modelled with the help of the Gaussian orthogonal ensemble (GOE) of random matrices [1]. In particular, the distribution of nearest-neighbour distances, the so-called number variance and spectral rigidity, but also spectral properties involving three- and four-point correlations could be explained in that way [2]. The GOE is appropriate if time-reversal (T) symmetry holds. In cases of broken T symmetry the Gaussian unitary ensemble (GUE) applies. Whereas the number of experimental realizations of GOE systems is numerous, until recently there was no experimental example for a spectrum of the GUE type. Then, 30 years after the formulation of random-matrix theory, two experimental realizations for which one expects to show the GUE spectral statistics were reported simultaneously [3, 4]. In both cases the spectra of microwave billiards are studied, and in both setups T invariance is broken by ferritic material inside the resonator. The two experiments differ, however, in one essential aspect. So *et al* [3] coated one wall of the resonator with ferrites and used the modified reflection properties to break T invariance. Stoffregen *et al* [4], on the other hand, attached a microwave by-pass with a built-in microwave isolator to the billiard such that the by-pass assumes unidirectional *absorption* properties which break the symmetry of time reversal. In both experiments quadratic level repulsion was found at small distances as is expected for the GUE. However, it is not obvious at all that the GUE can be applied

‖ On leave from Center for Theoretical Physics, Polish Academy of Sciences, Warszawa, Poland.

 

to systems containing strongly absorbing channels. But it was shown phenomenologically in [4] that a 'Hamiltonian' $H_{\mathrm{eff}} = H - \mathrm{i}\Gamma$ , where $H$ corresponds to the original billiard and where $\Gamma$ takes into account the unidirectional damping, can explain all but one of the experimental findings. Whereas in a Sinai billiard with broken symmetry quadratic level repulsion was found as expected, in the corresponding rectangular billiard linear level repulsion was observed, and this latter fact remained to be explained.

It is the purpose of this paper to tighten up the phenomenological arguments of [4] and to fill the gap mentioned. To this end a scattering-matrix approach is employed, as familiar from nuclear physics. The measuring antenna constitutes one scattering channel, whereas two further channels are given by the entrance and exit of the unidirectional transmission line. Absorption in the walls could be taken into account by further channels but is neglected here since its effect is quite small compared to the strong damping introduced in the unidirectional handle. It was shown recently that the scattering approach is very well suited to describing reflection and transmission properties of microwave billiards [5, 6]. In the present case, too, this approach will prove capable of quantitatively explaining all experimental findings, including the linear level repulsion in rectangular billiards with broken T symmetry.

## 2. The model

Before going into details we would like to stress one peculiarity of the experiment of [4] which we propose to analyse here. Time reversal invariance is broken by means of a unidirectional handle which supports only *one propagating mode* . The symmetry breaking perturbation is thus of *rank 1* (i.e. the symmetry breaking part is a projection operator on a one-dimensional subspace); on the other hand it must be strong. This is in contrast to standard models of symmetry breaking where the perturbation is of the same rank as the unperturbed matrix and can be small. Having this in mind we should be prepared to find results differing in several aspects from the predictions of the GUE or ensembles which interpolate between the GOE and the GUE [7].

Let us first look at a resonator with three waveguides attached each of which supports exactly one propagating mode. One of the waveguides serves to couple microwaves into and out of the resonator. The remaining two waveguides represent the entrance and exit ports of the attached unidirectional 'handle'. To 'form' the handle we shall eventually join the latter two parts.

Wave scattering in such a three-port system is described by a unitary $3 \times 3$ $S$-matrix of the structure [8]

$$S = I + 2\mathrm{i}W^{\dagger}\frac{1}{H - E - \mathrm{i}WW^{\dagger}}W \tag{2.1}$$

where $H$ is the Hamiltonian of the resonator, taken to be an $N \times N$ matrix, and $W$ an $N \times 3$ matrix. The three-column vectors $X_n$ ($n = 1, 2, 3$) of $W$ describe the coupling of the resonator to the three waveguides and are related to the values of the internal wavefunction at the coupling points. The three vectors in question need neither be normalized nor mutually orthogonal.

The $S$-matrix maps the incoming states in the waveguides into outgoing ones, $S|\mathrm{in}\rangle = |\mathrm{out}\rangle$. Let us label by 1 the waveguide (microwave cable) through which the measurement is performed. The waveguides 2 and 3 then represent the exit and entrance parts of the handle to the resonator. The wavefunctions $u_n$ inside the waveguides read

$$u_n(y) = a_n \mathrm{e}^{\mathrm{i}k_n y} + b_n \mathrm{e}^{-\mathrm{i}k_n y} \qquad n = 1, 2, 3 \tag{2.2}$$

with $y$ denoting the distance measured from the entrance point and $k_n$ being the corresponding wave vector. The $S$-matrix couples the vectors $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$,

$$\boldsymbol{b} = S\boldsymbol{a}. \tag{2.3}$$

The one-way character of the handle is imposed by coupling together the waveguides 2 and 3 and requiring

$$\begin{aligned} a_3 &= 0 \\ a_2 &= b_3. \end{aligned} \tag{2.4}$$

These properties mimic dissipation and imply non-unitarity of the $S$-matrix. Note that the dissipation inside the handle does not require the coefficient $b_2$ to be equal to zero. The waves entering the handle in the 'prohibited' direction and described by this coefficient are fully dissipated and do not play any role in the further consideration. Inserting the restrictions (2.4) into the input–output relation (2.3) we get the output $b_1$ through channel 1 in terms of the input $a_1$ as

$$b_1 = \tilde{S} a_1 \tag{2.5}$$

with the scattering coefficient

$$\tilde{S} = S_{1,1} + \frac{S_{1,2} S_{3,1}}{1 - S_{3,2}}. \tag{2.6}$$

Obviously, the scattering coefficient $\tilde{S}$ represents the reflection amplitude the modulus of which is measured in the experiment. With the help of the general representation (2.1) of the $S$-matrix we may rewrite $\tilde{S}$ as

$$\tilde{S} = 1 + X_1^\dagger [R + R X_2 (1 - S_{3,2})^{-1} X_3^\dagger R] X_1 \tag{2.7}$$

with $R$ the resolvent:

$$R = 2\mathrm{i}(H - E - \mathrm{i} W W^\dagger)^{-1}. \tag{2.8}$$

Decomposing

$$(1 - S_{3,2})^{-1} = \sum_{n=0}^{\infty} (S_{3,2})^n = \sum_{n=0}^{\infty} (X_3^\dagger R X_2)^n \tag{2.9}$$

and using the identity

$$X_2 (1 - S_{3,2})^{-1} X_3^\dagger = X_2 X_3^\dagger \sum_{n=0}^{\infty} (R X_2 X_3^\dagger)^n \tag{2.10}$$

we find

$$\begin{aligned} \tilde{S} &= 1 + X_1^\dagger \left[ R + R X_2 X_3^\dagger \sum_{n=0}^{\infty} \left( R X_2 X_3^\dagger \right)^n R \right] X_1 \\ &= 1 + X_1^\dagger \sum_{n=0}^{\infty} (R X_2 X_3^\dagger)^n R X_1 \\ &= 1 + X_1^\dagger (1 - R X_2 X_3^\dagger)^{-1} R X_1 \\ &= 1 + X_1^\dagger (R^{-1} - X_2 X_3^\dagger)^{-1} X_1. \end{aligned} \tag{2.11}$$

Upon writing out the resolvent $R$ we obtain our final form of the scattering coefficient,

$$\tilde{S} = 1 + 2\mathrm{i}X_1^\dagger \frac{1}{H_{\mathrm{eff}} - E - \mathrm{i}X_1X_1^\dagger} X_1 \tag{2.12}$$

with the non-Hermitian 'effective Hamiltonian'

$$H_{\mathrm{eff}} = H - \mathrm{i}(X_2 + X_3)(X_2 + X_3)^\dagger - \mathrm{i}(X_2X_3^\dagger - X_3X_2^\dagger) = H - \mathrm{i}P + Q. \tag{2.13}$$

Here we have introduced a Hermitian (non-normalized) projector $P = (X_2 + X_3)(X_2 + X_3)^\dagger$ as well as a second Hermitian operator $Q = -\mathrm{i}(X_2X_3^\dagger - X_3X_2^\dagger)$. We have not included $X_1X_1^\dagger$ in $H_{\mathrm{eff}}$ since the coupling of the antenna to the resonator is weak and only gives rise to small shifts and broadenings of the resonances [6] which can be neglected in the present context. It is the influence of the strongly coupled one-way handle which we are after, and that influence is accounted for in the effective Hamiltonian. We may identify the measured resonances with the eigenvalues of the operator $H_{\mathrm{eff}}$. The interaction term in $H_{\mathrm{eff}}$ was divided into two parts because of their physical distinction. The first part, $-\mathrm{i}P$, is non-Hermitian and thus primarily accounts for the absorption, in the ferritic material, of all waves entering the one-way handle in the forbidden direction. The second part, $Q$, is Hermitian and can therefore not be associated with the unidirectionality of the handle. Both interaction terms constitute strong perturbations of the Hamiltonian $H$ of the closed resonator; the absorptive part $-\mathrm{i}P$ must be strong since the complete or nearly complete absorption of waves entering the handle in the forbidden sense cannot be a small perturbation; the Hermitian part $Q$ has no reason to be small either since it owes its existence to the same two coupling vectors $X_2, X_3$, as does its non-Hermitian partner. At any rate, the absorptive coupling must be strong enough to bring about an appreciable change of the spectral statistics.

It is also important, at this point, to realize that upon traversing the handle in the allowed sense a wave picks up a phase shift. The coupling vectors $X_2, X_3$ must thus be taken as complex.

We have numerically diagonalized the effective Hamiltonian $H_{\mathrm{eff}}$ for two types of resonator Hamiltonians $H$, for one type drawing $N \times N$ matrices with $N = 100$ from the GOE (to simulate Sinai or other chaotic billiards) and for the other type drawing from the so-called Poissonian ensemble (PE) of diagonal matrices (to simulate rectangular or other regular billiards). These matrices $H$ were normalized such as to have the eigenvalues confined to the interval $[-2, 2]$, in the first case by choosing the radius in Wigner's semicircle law as 2, in the second case by working with independent diagonal elements covering the interval in question according to a box distribution or a Gaussian. To realize the interaction term in $H_{\mathrm{eff}}$ we chose the coupling vectors each with $N$ independent complex components with identical Gaussian distributions of zero mean and width $\sqrt{g/N}$,

$$\langle X_i \rangle = 0 \qquad \langle |X_j|^2 \rangle = g/N \qquad j = 2, 3 \tag{2.14}$$

thus introducing a coupling strength $g$. To justify such a choice we recall that the vector component $X_i$ is proportional to the value $\Psi(\boldsymbol{r}_0)$ of the internal wavefunction at the coupling point $\boldsymbol{r}_0$. For the case of nonintegrable (closed) billiards Gaussian behaviour of the wavefunction is predicted by the Gaussian ensembles. For integrable billiards, on the other hand, no such theoretical backing is available. We have therefore convinced ourselves that in that case results are qualitatively unchanged when the Gaussian distribution is replaced by another distribution (box, Poisson, or even delta in which later case no randomness at all is allowed).

For sufficiently large values of the coupling constant, in practice for $g$ larger than about 5, the $N$ eigenvalues of $H_{\mathrm{eff}}$ fall into two well separated clouds: one eigenvalue (the number

1 being due to the rank of the projector $P$) is nearly imaginary with its imaginary part near $-g$ while the remaining $N-1$ eigenvalues are nearly real, with imaginary parts of the order of $-1/g$ and their real parts covering the same interval as do the unperturbed eigenvalues of $H$. Clearly, it is the latter $N-1$ nearly real eigenvalues that we must take as representatives of the measured resonances, their real parts giving the locations and their imaginary parts giving the widths of the resonances. The large size of the coupling constant $g$ is justified by the analysis done in [6], which shows that under the experimental settings we are discussing here the appropriate value of $g$ around $g \approx 8$.

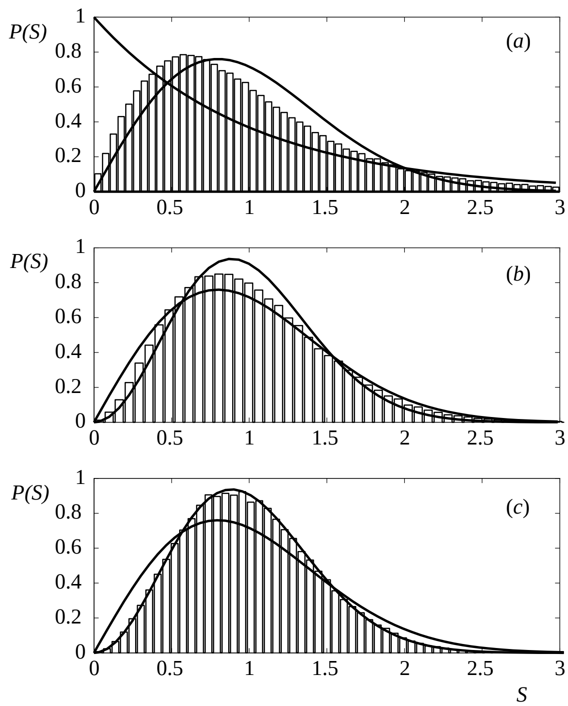


**Figure 1.** Distribution of resonance spacings $P(S)$ versus $S$ for the effective Hamiltonian $H_{\text{eff}}$ with $H$ taken from the Poisson ensemble ($a$), GOE ($b$) and GUE ($c$). The full curves correspond to the nearest-neighbour spacing distribution for the PE and GOE ($a$), and for the GOE and GUE ($b$) and ($c$).

The strong-coupling limit of large $g$ is indeed of relevance for us since only then do well-separated resonances arise, the strong damping of two 'modes' notwithstanding. It is then meaningful to investigate the distribution $P(S)$ of the spacings $S$ of nearest-neighbour resonances, after properly unfolding the spectrum to a constant mean density of resonances. The resulting spacing distributions were made smooth by collecting data from several thousand matrices $H_{\text{eff}}$ for each type. Much to our satisfaction the spacing distributions, depicted in figure 1, reveal linear repulsion for 'regular' resonator Hamiltonians $H$ and quadratic repulsion in the 'chaotic' case, i.e. for $H$ drawn from the GOE. We have also admitted resonator Hamiltonians from the GUE and found that the quadratic repulsion characteristic for the GUE remains for the matrices $H_{\text{eff}}$, as also shown in figure 1. The figure shows that the spacing distribution found for $H$ from the PE differs from that of the GOE for large spacings; the similarity does not go much beyond the linearity of $P(S)$

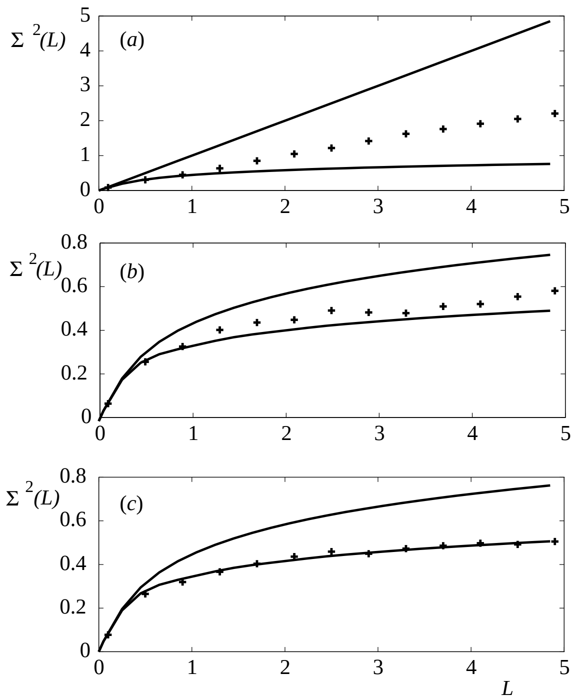


**Figure 2.** Number variance $\Sigma^2(L)$ versus $L$ for the effective Hamiltonian $H_{\mathrm{eff}}$ with $H$ taken from the PE ($a$), the GOE ($b$), and the GUE ($c$). The full curves correspond to the theoretical curves for the PE and GOE ($b$), and for the GOE and GUE ($b$) and ($c$).

for small $S$; analogously, the $P(S)$ arising for $H$ from the GOE resembles closely that of the GUE only for small spacings. We would like to emphasize that all of our results for $P(S)$ are insensitive to the precise value of the coupling strength $g$, provided that strength is larger than about 5.

For the number variance $\Sigma^2$ shown in figure 2 the situation is analogous. Here the additional channel has the effect that for $H$ taken from PE or GOE, the number variance $\Sigma^2(L)$ follows closely the GOE and GUE curves, respectively, for $L < 1$. For larger $L$ values, deviations of $\Sigma^2(L)$ towards PE and GOE behaviour, respectively, is found. Again the additional channel has no influence, if $H$ is taken from the GUE.

In all these respects our numerical findings agree with the experimental results of [4]. The breaking of time reversal invariance by a one-way handle, modelled here by a non-Hermitian perturbation of rank 2 in $H_{\mathrm{eff}}$, increases the degree of level repulsion from linear to quadratic if enforced on a chaotic billiard; if the unperturbed billiard is regular, the one-way handle brings about linear repulsion, i.e. again increases the degree of repulsion by 1. Nonuniversal behaviours of $P(S)$ for large spacings, much as those seen in figure 1, were also observed experimentally. This behaviour may be blamed on the low rank of the perturbation breaking the symmetry. Upon increasing the number of channels supported by the handle and working with a Sinai-type unperturbed resonator we would expect to incur a spacing distribution approaching the universal GUE curve.

Before deepening the comparison with experiments we propose to pause for an intuitive explanation of our numerical findings on symmetry breaking in the following two sections.

## 3. Three-pole approximations

Since the interaction parts $P$ and $Q$ of the effective Hamiltonian $H_{\rm eff}$ are of rank 1 and 2, respectively, it is convenient to evaluate the eigenvalues of $H_{\rm eff}$ using resolvent equations. For the resolvent $R_{\rm eff}(E) = (H_{\rm eff} - E)^{-1}$ of $H_{\rm eff}$ we have

$$R_{\rm eff}(E) = \frac{1}{1 + (-\mathrm{i}P + Q)R(E)} R(E) \tag{3.1}$$

with $R(E) = (H - E)^{-1}$. Since the poles of $R_{\rm eff}(E)$ coincide with the eigenvalues of $H_{\rm eff}$ the eigenvalue problem for $H_{\rm eff}$ is equivalent to solving the equation

$$(-\mathrm{i}P + Q)R(E) f = -f. \tag{3.2}$$

In other words, the parameter $E$ must be chosen such that the operator $(-\mathrm{i}P + Q)R(E)$ has an eigenvector $f$ with eigenvalue$-1$. At this point the low ranks of $P$ and $Q$ come in helpful: the vector $f$ is spanned by the two vectors $X_2$ and $X_3$, whereupon we are led to diagonalizing a $2 \times 2$ matrix.

Moreover, in the strong-coupling limit of interest to us the components of the vectors $X_2, X_3$ are typically large such that the matrix elements of the interaction $-\mathrm{i}P + Q$ are of the order $g \gg 1$. The eigenvalue equation (3.2) thus simplifies to

$$(-\mathrm{i}P + Q)R(E) f = 0. \tag{3.3}$$

The roots $E$ of the foregoing equation can be identified with the real parts of those $N - 1$ eigenvalues of $H_{\rm eff}$ which lie close to the real axis and thus give the measured resonances.

We propose to discuss the influence of the perturbations $-\mathrm{i}P$ and $Q$ on the spectrum of $H_{\rm eff}$ separately. Let us start with the dissipative term $-\mathrm{i}P = -\mathrm{i}(X_2 + X_3)(X_2 + X_3)^\dagger$ which as already mentioned is of rank 1. The vector $f$ in question must thus be proportional to $(X_2 + X_3)$. Using this observation we may rewrite the equation $PR(E)f = 0$ in the eigenrepresentation of the unperturbed resonator Hamiltonian $H$ as the algebraic equation

$$F(E) = \sum_n \frac{|X_2^{(n)} + X_3^{(n)}|^2}{E_n - E} = 0 \tag{3.4}$$

where $X_j^{(n)}$ are the components of the vector $X_j$, $X_j = (X_j^{(1)}, \ldots, X_j^{(N)})$, and the $E_n$ are the eigenvalues of $H$. Clearly, the function $F(E)$ is monotone meromorphic with poles located at $E_n$. Between two poles there is exactly one zero of $F(E)$. Moreover, if three poles cluster together the two zeros locked up in between come close to each other. To analyse the distribution $P(S)$ of nearest-neighbour spacings $S$ for small $S$ it is enough to restrict the sum $F(E)$ in (3.4) to only three terms, those whose poles lock up the two colliding zeros of $F(E)$. Then (3.4) becomes a quadratic equation the zeros of which are real and must coincide for a crossing of two resonances. To write out the separation of the two roots of the quadratic equation in question we set, without loss of generality, the zero of energy on the middle one of the three poles of $F$, $E_1 = 0$, such that of the neighbouring poles one, say $E_0$, is negative while the other one, $E_2$, is positive; to further facilitate the notation we write $w_n = |X_2^{(n)} + X_3^{(n)}|^2$ for the weights of the three poles in consideration and set, again without loss of generality, $w_1 = 1$. The separation $\Delta$ of the two roots is then given by

$$4(1 + w_0 + w_2)\Delta^2 = (E_2(1 + w_0) + E_0(1 + w_2))^2 - E_0E_2(1 + w_0 + w_2). \tag{3.5}$$

Since $E_0E_2 \leqslant 0$, $\Delta^2$ is manifestly non-negative as it must be. To enforce a crossing of resonances, $\Delta = 0$, one must necessarily have $E_0 = E_2 = 0$, i.e. a coincidence of next-nearest neighbours among the levels of the unperturbed resonator Hamiltonian. This observation immediately allows one to establish the degree of repulsion between the resonances through the so-called codimension of the crossing, i.e. the number of real parameters necessary to enforce the crossing: the degree of repulsion is smaller by 1 than the codimension [9]. For a regular resonator as, e.g., one of rectangular shape (apart from the attached handle) the unperturbed eigenvalues $E_n$ have Poissonian statistics, i.e. are independent of one another and have no inhibitions to cross; two real parameters then suffice to induce a collision of three neighbours, $E_0 = E_1 = E_2$; the two resonances locked up in between them are then also degenerate; we are thus led to expect linear repulsion between the resonances, i.e. the near-real eigenvalues of $H_{\text{eff}}$. On the other hand, if the unperturbed Hamiltonian $H$ is a typical member of the GOE, as is the case for resonators of the Sinai type, the codimension of a triple degeneracy is 3 from which fact we infer quadratic repulsion of our resonances.

Since it may not be widely known that for real symmetric matrices (of which a GOE matrix is an example) three parameters are needed to enforce a triple degeneracy in its spectrum, it may be well to point out that this question may be decided, in the sense of perturbation theory of nearly degenerate levels, by studying real symmetric $3 \times 3$ matrices. To arrange a coincidence of the three real roots of the cubic secular equation of such a matrix one indeed need three controllable real parameters.

To conclude the investigations of this section we turn to showing that the Hermitian term $Q$ in the interaction part of $H_{\text{eff}}$ cannot change the behaviour induced by the dissipative term. In particular, in spite of being complex Hermitian and having matrix elements of the same order of magnitude as its dissipative partner just discussed, the operator $Q$ by itself could not bring about quadratic level repulsion in the chaotic case, due to its being of rank 2. To see this we must discuss the equation $QR(E)f = 0$ . The vector $f$ may now be sought as a linear combination of the two vectors $X_2$ and $X_3$; this is most conveniently done with the help of the projection ansatz $f = (X_2X_2^\dagger - X_3X_3^\dagger)h$ with $h$ an arbitrary vector. By again employing the eigenrepresentation of the unperturbed resonator Hamiltonian $H$ we now find the equation

$$\sum_{n\neq m} \frac{|X_2^{(n)}|^2|X_3^{(m)}|^2 - X_2^{(n)}X_3^{(n)*}X_2^{(m)*}X_3^{(m)}}{(E_n - E)(E_m - E)} = 0. \tag{3.6}$$

To investigate the spacing distribution of the roots of the foregoing equation for small $S$ we restrict once more the sum to one over three neighbouring poles. But we are then immediately led to a *linear* equation in $E$. Instead of locking up two roots in between themselves the three neighbouring poles are now associated with a single root and hence are impotent to change the degree of repulsion between the roots. These findings are supported also by numerical calculations in which we only took into account either $P$ or $Q$.

## 4. Approximation by $2 \times 2$ matrices

According to a well known surmise of Wigner's, the distribution of nearest-neighbour spacings of large matrices from any of the classic Gaussian and circular ensembles are extremely closely approximated by the simple spacing distributions of the corresponding ensembles of $2 \times 2$ matrices. It is therefore tempting to ask whether the degree of repulsion between the near-real eigenvalues of our non-Hermitian operator $H_{\text{eff}}$ can also be understood by working with suitable $2 \times 2$ matrices. Here we propose to sketch a positive answer which will nicely complement the intuitive arguments of the previous section and the numerical results of section 2. For the sake of simplicity we shall confine ourselves to

$$H_{\text{eff}} = H - \mathrm{i}gP \qquad P = XX^\dagger \qquad X^\dagger X = 1 \tag{4.1}$$

with $H$ from either the Gaussian orthogonal or the Poissonian ensemble of $N \times N$ matrices and $X$ a complex $N$-component vector. We thus exclude the complex Hermitian rank 2 perturbation which is unable to influence the degree of repulsion anyway. Note that we have now imposed normalization to unity on the vector $X$ and therefore incur a coupling constant $g$ which was previously hidden in the norm of $X$.

In the strong-coupling limit of interest here, $g \gg 1$, it is not inappropriate to consider the original resonator Hamiltonian as a 'small perturbation' of the damping term $-\mathrm{i}gP$. Adopting that view one confronts, in zero order with respect to $H$, one large negative imaginary eigenvalue, $-\mathrm{i}g$, of $H_{\mathrm{eff}}$ and the $(N-1)$-fold degenerate eigenvalue zero, just because $P$ is of rank 1. In the sense of degenerate-state perturbation theory the most important effect of $H$ on $H_{\mathrm{eff}}$ is then to lift the latter degeneracy and thus to provide the cloud of $N-1$ near real eigenvalues which constitute the measurable resonances. We may determine those $N-1$ eigenvalues in lowest order in $H$ by diagonalizing $H$ in the $(N-1)$-dimensional null space of $P$. That space is formed by $N-1$ vectors of $N$ components which are orthogonal to $X$ but otherwise arbitrary.

Let us focus our attention on the $(N-1) \times (N-1)$ matrix $H$ thus incurred, and here in particular on two of its eigenvalues which are nearest neighbours of one another and have a distance smaller than the distance of either to any third eigenvalue. We may then once more invoke perturbation theory for nearly degenerate levels and consider a $2 \times 2$ representative of $H$ formed with suitable approximants to the corresponding two eigenvectors of $H$ chosen from the $N-1$ vectors orthogonal to $X$. We may choose two such approximants by taking two orthonormal eigenvectors $\xi_1, \xi_2$ of the original $N \times N$ matrix $H$ whose eigenvalues $E_1$ and $E_2$ are nearest neighbours and orthogonalize these on $X$ to obtain $\tilde{\xi}_i = \xi_i - X^{(i)*}X$ where $X^{(i)}$ again denotes the $i$th component of $X$ in the eigenbasis of $H$. The $2 \times 2$ representative of $H$ in the null space of $P$ thus obtained is easily diagonalized and yields two eigenvalues $\tilde{E}_i$ separated by

$$\tilde{E}_1 - \tilde{E}_2 = (1 - |X^{(1)}|^2 - |X^{(2)}|^2)^{-1} \sqrt{\sum_{i,j=1}^{2} \Lambda_{ij}(E_i - X^\dagger H X)(E_j - X^\dagger H X)} \tag{4.2}$$

with a $2 \times 2$ matrix $\Lambda$ which must be and is in fact easily checked to be non-negative. We conclude that to enforce a crossing of the two resonances, $\tilde{E}_1 - \tilde{E}_2 = 0$, we need to have $E_1 = E_2 = X^\dagger H X$. The codimension of a crossing of resonances is thus larger by 1 than the codimension of a crossing of the corresponding pair of original levels since both $E_1$ and $E_2$ must be steered to coincide with the expectation value of $H$ in the state $X$. We are thereby again led to linear repulsion for an originally regular $H$ and to quadratic repulsion in the GOE case.

Having seen the degree of level repulsion to increase by 1 for $H$ both from the Poissonian and the Gaussian orthogonal ensemble one might be tempted to rush to inappropriate extrapolation for the GUE. To rid oneself of such temptation one should, for the case of $H$ from the GUE, consider $H$ in the eigenrepresentation of $P$ where it is a complex Hermitian $N \times N$ matrix; upon discarding the first row and the first column one encounters an $(N-1) \times (N-1)$ submatrix which represents the operator $H$ in the null space of $P$; but that latter matrix is as much a typical member of the GUE of $(N-1) \times (N-1)$ matrices as is the original matrix of the GUE of $N \times N$ matrices. This means that in the strong-coupling limit the presence of the dissipative term $\mathrm{i}P$ does not change the spectral statistics of the GUE ensemble—see figure 1($c$) for numerical results. On the other hand, when the original $H$ is from the GOE or the Poissonian ensemble, the transformation to the eigenrepresentation of $P$ does change the statistical properties of the $(N-1) \times (N-1)$

submatrix such that the above considerations are necessary to redetermine the degree of level repulsion.

Unfortunately, the expression (4.2) for the spacing of neighbouring resonances is not sufficiently simple to suggest a way towards a closed form for the full spacing distribution for the cloud of resonances of $H_{\rm eff}$.

## 5. Experimental results

We can now proceed to a more detailed comparison of our results with the experimental data. First, we recall from the previous two sections that the perturbation of the billiard by the one-way handle leads to a degree of level repulsion as experimentally observed but, due to its low rank, gives spacing distributions not otherwise quantitatively agreeing with the observed distributions. Rather, for spacings not small compared to the mean one, the distribution we find for our model matrices, like the experimentally observed ones, resemble the distributions characteristic for the unperturbed billiards (see figure 1). More specifically, for the rectangular resonator with the handle attached $P(S)$ displays linear repulsion for small $S$ but decays exponentially in the wing; for the Sinai billiard with the handle attached, the broken symmetry entails quadratic repulsion for small $S$ as for the GUE but returns to resembling GOE behaviour in the wing. By the same reasoning one is led to expect the number variance $\Sigma^2(L)$ to reflect symmetry breaking for small $L$ but to cross over to the case of preserved symmetry as the interval length $L$ increase (see figure 2).

For a careful comparison of the theoretical and experimental spacing distributions $P(S)$ we must first mention that not all levels (resonances) have been detected in the experiment. The fractional amount of lost levels in the individual spectra is as high as 0.39 for the asymmetric Sinai and 0.49 for the rectangular resonator [4]. Such deplorably large loss is in contrast with previous measurements on resonators with time reversal invariance [11] for which the fractional loss of levels amounted to merely about 0.07 and 0.14 for the Sinai and rectangular billiards, respectively. The increased loss can be explained simply: the unidirectional handle entails resonance broadening and thus a larger number of unresolved resonances.

We have found it reasonable to divide the missing levels into two groups. The ones from the first group did not show up in the experiment because the measuring antenna was placed on a nodal line of the corresponding wavefunction. The fraction of such miss-outs cannot be larger than, and should be given approximately by the amount of levels overlooked in experiments performed without the one-way handle, i.e. 0.07 for the asymmetric Sinai and 0.14 for the rectangular resonators. We assume that these missing levels are randomly distributed among all levels of the system. In the model calculations the corresponding fraction of randomly chosen levels was discarded.

The second group are levels overlooked due to level broadening. This group was taken care of by discarding all levels numerically found for the model whose width was larger then a certain part of the distance between neighbouring resonances, $|\Im(\lambda_n)| > A\times\min(|\Re[\lambda_{n+1}-\lambda_n)|, |\Re(\lambda_n-\lambda_{n-1})]|$. The constant $A$ was chosen such that the combined loss due to both groups of omitted levels coincided with the aforementioned loss reported by the experimenters. It is worth mentioning that in both cases (Sinai and rectangular resonator) the value of $A$ was found to be approximately the same, $A \sim 0.1$. Numerical calculations show that the exact choice of the parameter $A$ does not have a substantial influence on the character of the resulting spacing distribution, provided $A$ is not taken to be too small.

The results obtained for nearest-neighbour spacing distribution and number variance are plotted on figures 3 and 4, respectively. The agreement with the experimental data is now

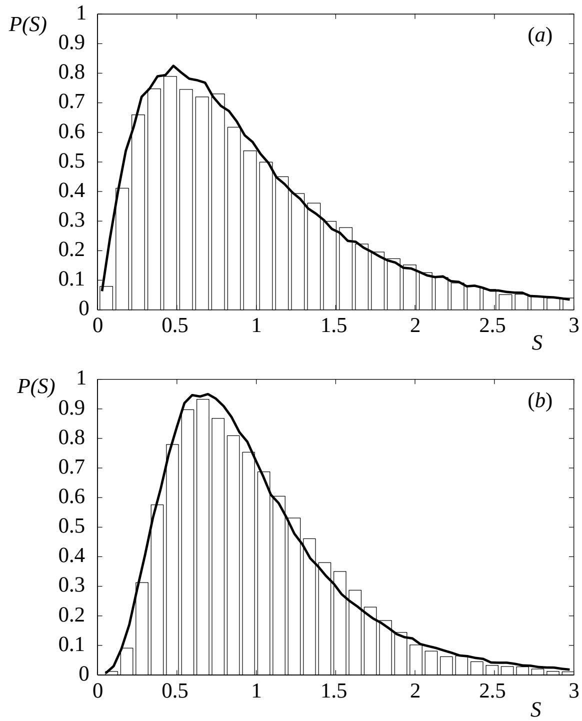


**Figure 3.** Experimentally obtained spacing distribution $P(S)$ versus $S$ (bins) compared with results of our model calculation. ($a$) Rectangular resonator (total fraction of missing levels 49%, fraction of randomly discarded levels 14%). ($b$) Asymmetric Sinai resonator (total fraction of missing levels 39%, fraction of randomly discarded levels 7%).

very satisfactory indeed. The small deviations seen in figure 4 between the theoretical and the experimental number variance increases more and more with increasing $L$. For $L \geqslant 8$ the experimental loss makes a reliable determination of $\Sigma^2(L)$ impossible.

## 6. Conclusion and outlook

Our model involves a $3 \times 3$ $S$-matrix, with one channel accounting for the weak coupling of the antenna and the remaining two for the strong coupling of the one-way handle. It is most satisfactory to see the fine quantitative agreement of the level spacing distributions of the microwave experiment and the model.

An interesting by-product of the arguments of sections 3 and 4 is that of the two terms contributing to the effective Hamiltonian $H_{\mathrm{eff}}$, the Hermitian one, $Q$, in spite of being complex, is incapable by itself of increasing the degree of repulsion due to its low rank. While the dissipative term $-\mathrm{i}P$ has even lower rank, it is in fact responsible for the modification of the spectral statistics since it is non-Hermitian, i.e. dissipative.

An immediate consequence arises for possible future experiments. Instead of attaching a one-way handle one might attach a single channel ending either in a wave sink or just opening up to the outside world like a trumpet. Just like a musical instrument such a microwave trumpet should have well defined resonances and yet display broken time reversal in the resonance statistics, due to the strongly coupled horn serving as an exit port.

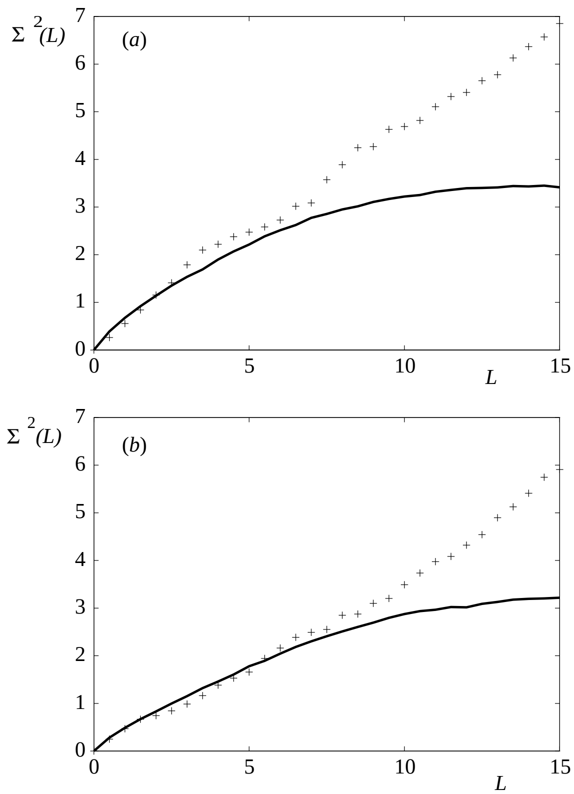


**Figure 4.** Number variance $\Sigma^2(L)$ versus $L$ evaluated from experimental data (marks). ($a$) For rectangular resonator, compared with prediction for GOE with randomly disregarded levels. ($b$) For Sinai resonator, compared with prediction for GUE with randomly disregarded levels. Total amount of discarded levels same as in the previous figure.

A further remark is indicated in view of the experiment reported in [3]. There the time reversal symmetry was broken by covering a whole wall of the resonator with ferritic material. Inasmuch as an $S$-matrix approach like the one advocated here might apply one is tempted to surmise that many more channels would be involved rather than just one or two. Consequently, the pertinent spacing distribution as well as other spectral statistics should be more faithful, in the overall behaviour, to the GUE than for the experiment on Sinai shaped resonators of [4].

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